A Capacity Scaling Procedure for the Capacitated Network Design Problem with Piecewise Linear Costs

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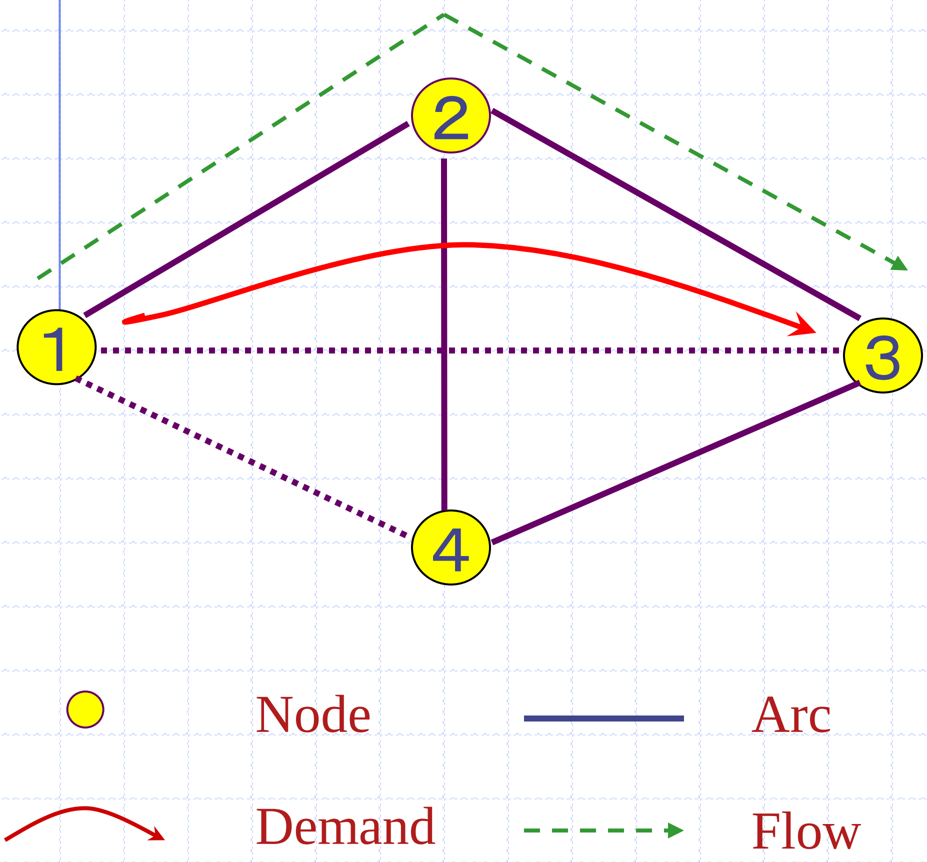
Introduction

The multi-commodity capacitated network design problem with piecewise linear costs

- Basic, strong and extended formulation
- Capacity scaling procedure

- Demand of multi-commodities
- Arc capacities
- Piecewise linear flow cost function
- The total flow costs is minimized
- The design of the network and paths of the multi-commodity flows is found

Multi-commodity network design model



- **Node**:distribution center
- **Arc**: less-than-truckload service or a feeder service
- **Demand**: freight from origin to destination
- **Flow**: freight flow from origin to destination

Literatures

- Crainic–Gendron–Hernu（2003）: Slope scaling methods, changing the flow cost with a Lagrange relaxation method and long term memories for capacitated network design problem.
- Chen-Katayama-Kubo(2005):Capacity scaling methods for capacitated network design problem with a strong formulation.
- Croxton-Gendron-Magnanti(2004): Basic, strong and an extended formulation for capacitated network design problem with piecewise linear costs

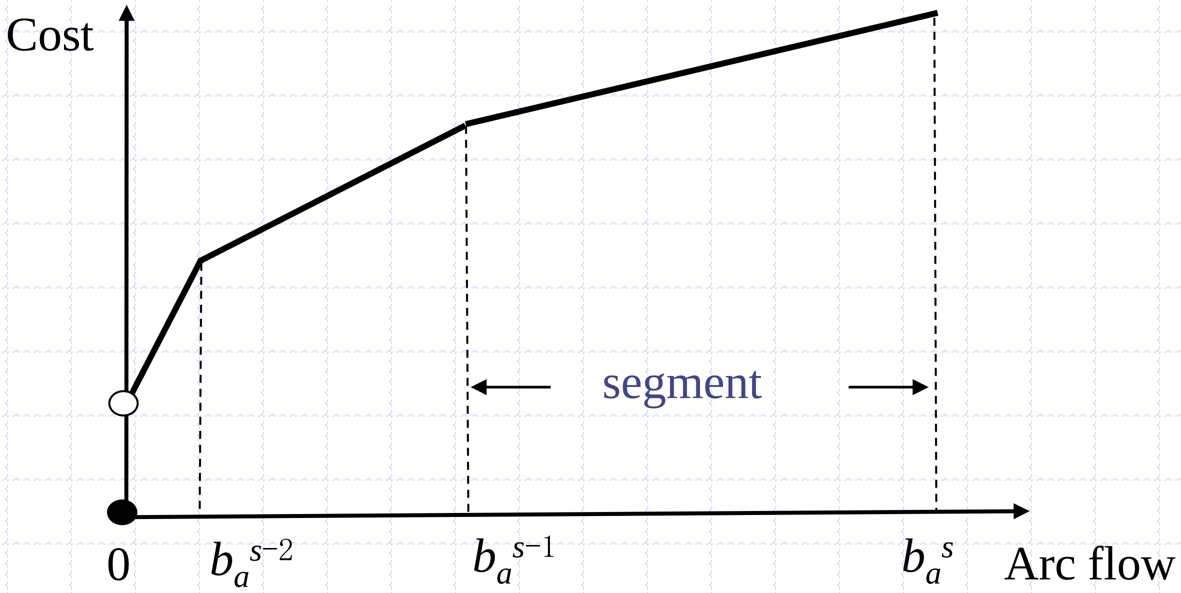
Notations

- N ： set of nodes
- A ： set of arcs
- A_n^{+} ： set of arcs, which go in from another node
- A_n^{-} ： set of arcs, which go out to another node
- K ： set of commodities
- S_a ： set of segments in *a* piecewise linear function
- O^k ： origin node for commodity k
- D^k ： destination node for commodity k

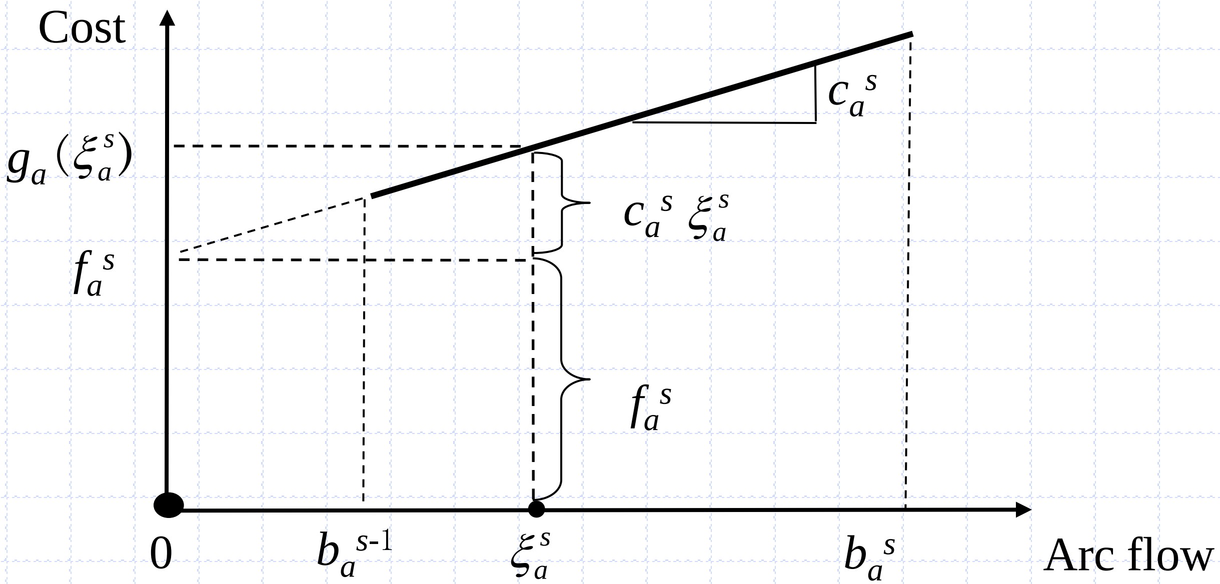
Notations

- d^k : demand of commodity k
- $\boldsymbol{X_a}$: flow variable on the arc a
- $\xi_{\boldsymbol{a}}^{\boldsymbol{s}}$: flow variable in segment s on arc a
- x_a^k : flow variable on arc a of commodity k
- y^s_a : design variable of segment s on arc a
- $\boldsymbol{g_a(X_a)}$: flow cost function on arc a

Piecewise linear cost function



Segment of a piecewise linear function



- If a flow lies in segment *s*, $\xi_a^s = arc\ flow\ X_a$,otherwise $\xi_a^s = 0$
- Flow cost in segmets *s* : $g(\xi_a^s) = c_a^s \xi_a^s + f_a^s y_a^s$

Basic model (*PLCB*)

Croxton et al.(2004)

variable cost

fixed cost

(*PLCB*) min $$\sum_{a\in A}\sum_{s\in S_a}(c_a^s\xi_a^s + f_a^s y_a^s) \quad (1)$$

s.t. $$\sum_{a\in A_n^+} x_a^k - \sum_{a\in A_n^-} x_a^k = \begin{cases} -d^k & if\ \ n = O^k \\ d^k & if\ \ n = D^k \\ 0 & otherwise \end{cases} \quad n\in N, \quad k\in K \quad (2)$$

conservative equation

$$X_a = \sum_{s\in S_a}\xi_a^s \qquad a\in A \quad (3)$$

$$X_a = \sum_{k\in K} x_a^k \qquad a\in A \quad (4)$$

the total flow on arc *a* lies between a lower and upper bound of a segment

$$b_a^{s-1} y_a^s \le \xi_a^s \le b_a^s y_a^s \qquad a \in A, \quad s \in S_a \qquad (5)$$

the sum of design variables on arc *a* = 0 or 1

$$\sum_{s \in S_a} y_a^s \le 1 \qquad a \in A \qquad (6)$$

$$x_a^k \ge 0 \qquad a \in A, \quad k \in K \qquad (7)$$

$$y_a^s \in \{0,1\} \qquad a \in A, \quad s \in S_a \qquad (8)$$

Strong model (*PLCS*)

Croxton et al.(2004)

(*PLCS*) min $$\sum_{a\in A}\sum_{s\in S_a}(c_a^s \xi_a^s + f_a^s y_a^s)$$

s.t. $$x_a^k \le d^k \sum_{s\in S_a} y_a^s \quad a\in A, \ k\in K \qquad (9)$$

(2), (3), (4), (5), (6), (7), (8)

When arc *a* has no flow, the sum of design variable = 0
and the flow of each commodity = 0
When arc *a* carries flow, the sum of design variable = 1
and flow of commodity $k <= d^k$

Extend model (*PLCE*)

- ζ_a^{ks}: extend flow variables for commodity *k*, segment *s* and arc *a*

$$\zeta_a^{ks} = \begin{cases} x_a^k & \text{When the total flow on the arc } a \text{ lies in segment } s \\ 0 & \text{Otherwise} \end{cases}$$

disaggregate for section *s* of (9)

$$\zeta_a^{ks} \le d^k y_a^s \qquad a \in A, \qquad k \in K, \quad s \in S_a \qquad (10)$$

$$x_a^k = \sum_{s \in S_a} \zeta_a^{ks}, \quad \xi_a^s = \sum_{k \in K} \zeta_a^{ks}$$

Extend model (*PLCE*)

Croxton et al.(2004)

$$(PLCE) \quad \min \sum_{a\in A}\sum_{s\in S_a}(c_a^s \sum_{k\in K}\zeta_a^{ks} + f_a^s y_a^s)$$

extend variable

$$\sum_{a\in A_n^+}\sum_{s\in S_a}\zeta_a^{ks} - \sum_{a\in A_n^-}\sum_{s\in S_a}\zeta_a^{ks} = \begin{cases} -d^k & if \ \ n = O^k \\ d^k & if \ \ n = D^k \\ 0 & otherwise \end{cases}$$

$$X_a = \sum_{k\in K}\sum_{s\in S_a}\zeta_a^{ks} \qquad a \in A$$

$$b_a^{s-1} y_a^s \le \sum_{k\in K}\zeta_a^{ks} \le b_a^s y_a^s \qquad a \in A, \quad s \in S_a$$

$$\sum_{s\in S_a} y_a^s \le 1 \qquad a \in A$$

$$y_a^s \in \{0,1\} \qquad a \in A, \quad s \in S_a$$

extended forcing constraints disaggregated for both commodity *k* and segment *s*

$$0 \le \zeta_a^{ks} \le d^k y_a^s \qquad a \in A \qquad k \in K \qquad s \in S_a$$

Capacity scaling procedure

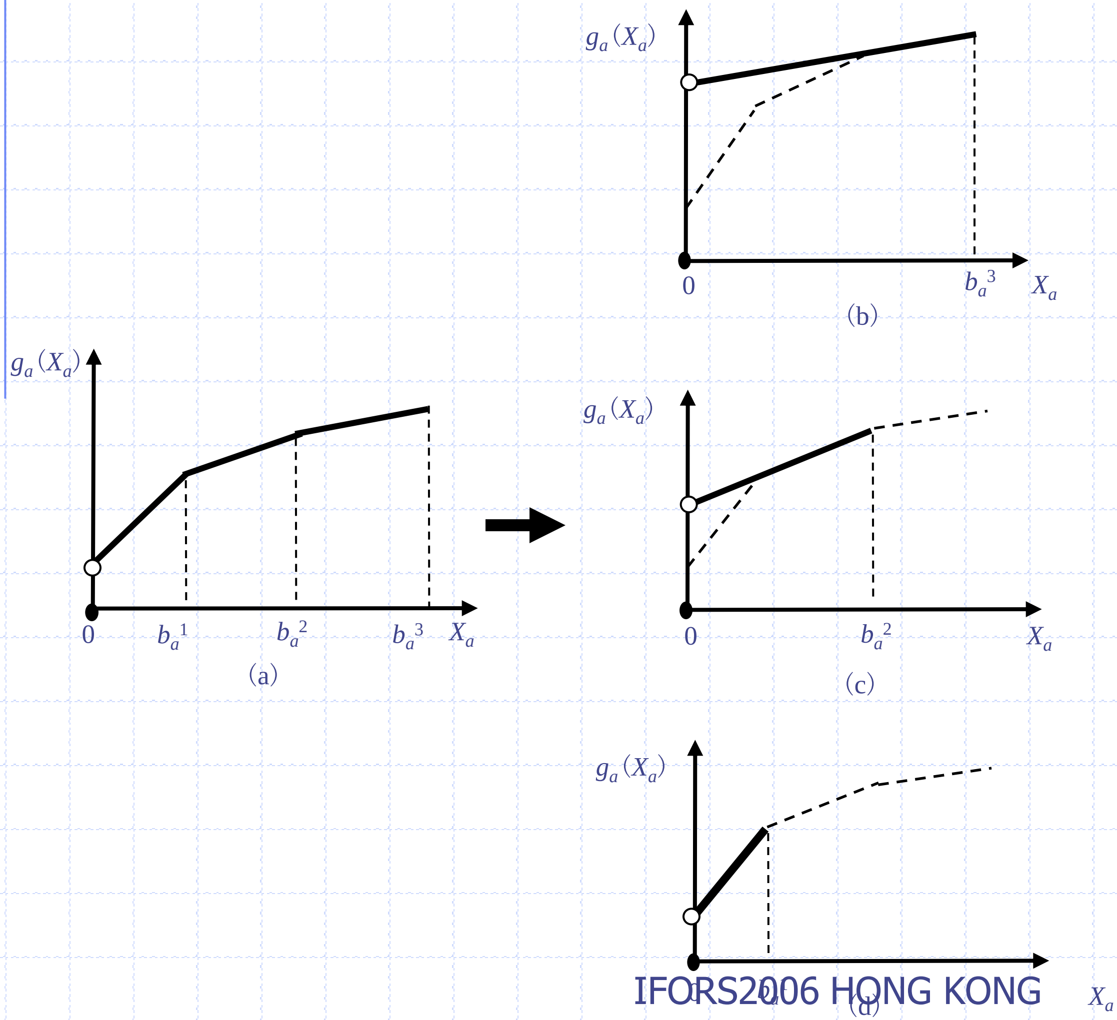
- *PLCB*, *PLCS* and *PLCE* are mixed integer programming problems
- It is difficult to solve these problems by mathematical programming software directly.
- We present approximate methods with a capacity scaling procedure.

Capacity scaling procedure

- Arc capacities change based on flow solutions of a linear relaxation problem.
- The linear relaxation problem is solved.
- Above procures are repeated.
- Approximate solutions are derived from relaxed solutions

Capacity scaling procedure

Each segment of arc *a* → one arc



segment 3

Lower bound =0

capacity of arc3 = b_a^3

segment 2

Lower bound =0

capacity of arc2= b_a^2

segment 1

Lower bound =0

capacity of arc1 = b_a^1

Capacity scaling procedure for *PLCB*

- Change to the capacitated problem

$$b_a^{s-1} y_a^s \le \xi_a^s \le b_a^s y_a^s \quad \longrightarrow \quad 0 \le \xi_a^s \le b_a y_a^s \quad y_a^s \in \{0,1\}$$

- Linear relaxation for variable y^s_a

$$y_a^s \in \{0,1\} \quad \rightarrow \quad 0 \le y_a^s \le 1$$

- Solve LP

$\tilde{y}_a^s$: optimal design variable solution of LP

$\tilde{\xi}_a^s$: optimal flow variable solution of LP

Capacity scaling procedure for *PLCB*

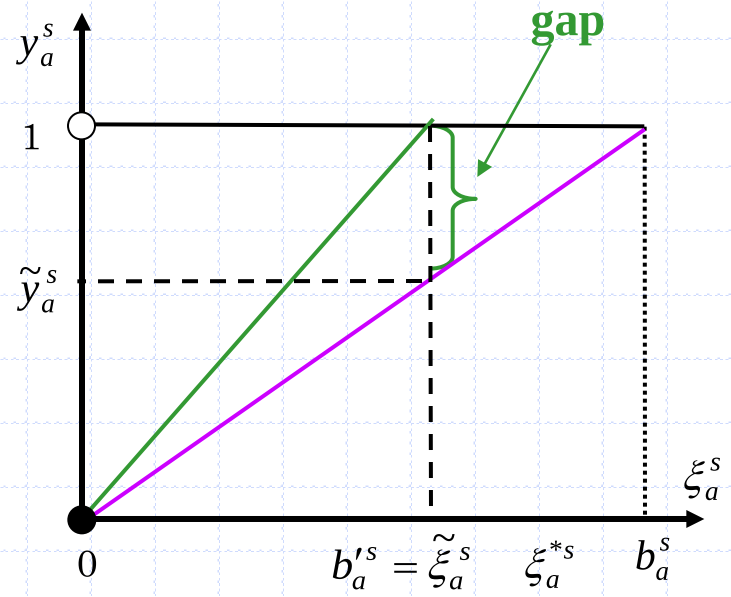
- Change capacities

$$\tilde{\xi}_a^s \le b_a^s \tilde{y}_a^s \;\rightarrow\; \tilde{\xi}_a^s = b_a^s \tilde{y}_a^s \xrightarrow{y_a^s=1} b'^{s}_{s} := \tilde{\xi}_a^s$$

- $\tilde{y}_a^s <= 1 \rightarrow$

  $\tilde{\xi}_a^s, b'^s_a$ decrease monotonically

- Once b^{*s}_a <= the optimal flow of *PLCB*, the optimal solution can never be obtained.



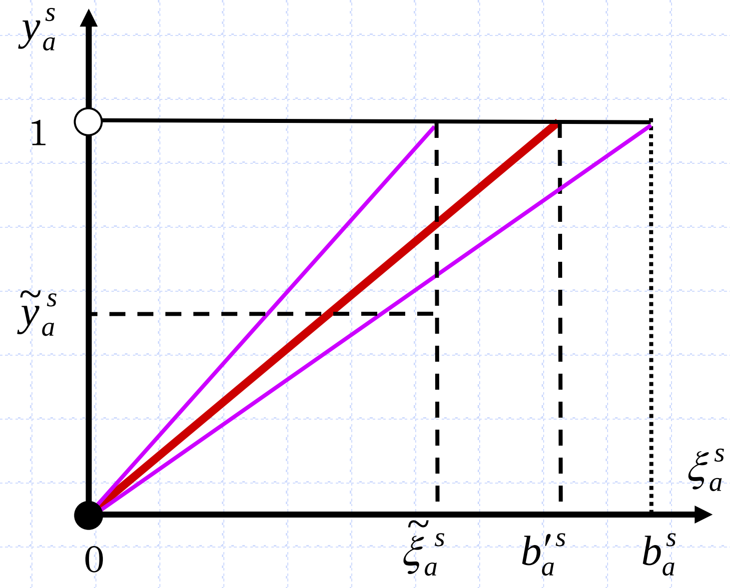
Capacity scaling procedure for *PLCB*

- Change capacity by the smoothing parameter λ

$$b'^{s}_{a} := \lambda \tilde{\xi}^{s}_{a} + (1-\lambda) b'^{s}_{a}$$

↓

$$b'^{s}_{a} := \lambda b'^{s}_{a}\, y^{s}_{a} + (1-\lambda) b'^{s}_{a}$$



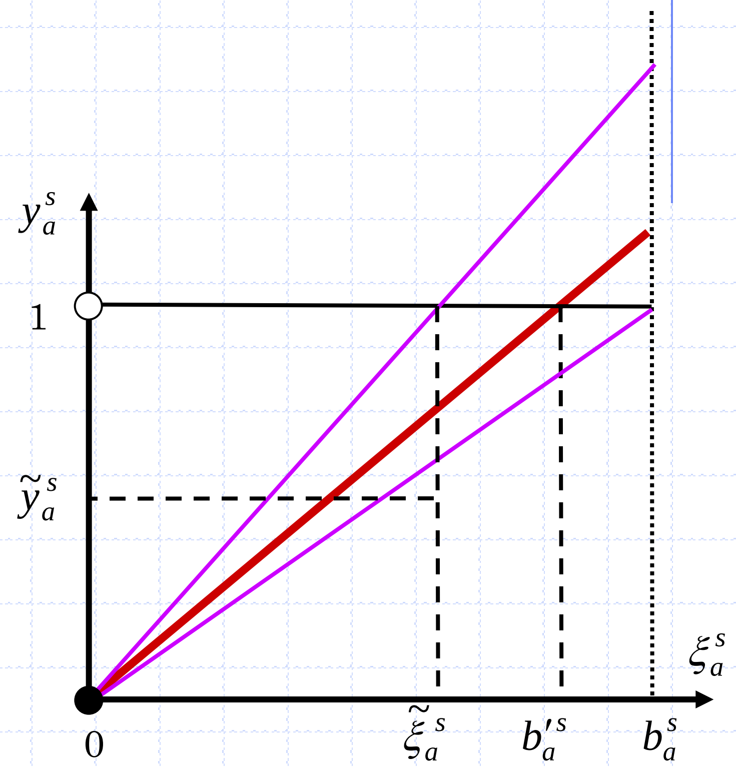
  - Prevent large variations of capacities

Capacity scaling procedure for *PLCB*

- Change the upper bound of y^s_a

$$0 \le y^s_a \le 1 \quad \Rightarrow \quad 0 \le y^s_a \le \frac{b^s_a}{b'^s_a}$$

  - $\tilde{\xi}^s_a$ can get be greater than b'^s_a
  - Prevent b^s_a > capacity $\tilde{\xi}^s_a$



- Change coefficients of y^s_a

$$\sum_{s \in S_a} y^s_a \le 1 \quad a \in A \quad (6) \quad \Rightarrow \quad \sum_{s \in S_a} (b'^s_a / b^s_a) y^s_a \le 1 \quad a \in A$$

$PLCBL(\boldsymbol{b}')$

Linear relaxation of *PLCBL* with capacity $\boldsymbol{b}' = (b'^{s}_{a})$

PLCBL(b')

$$\min \sum_{a\in A}\sum_{s\in S_a}(c^s_a \xi^s_a + f^s_a y^s_a)$$

$$\sum_{a\in A^+_n} x^k_a - \sum_{a\in A^-_n} x^k_a = \begin{cases} -d^k & \text{if } n = O^k \\ d^k & \text{if } n = D^k \\ 0 & \text{otherwise} \end{cases} \quad n\in N,\quad k\in K$$

$$X_a = \sum_{k\in K} x^k_a \qquad a\in A$$

$$X_a = \sum_{s\in S_a} \xi^s_a \qquad a\in A$$

$$0 \le \xi^s_a \le b'^s_a y^s_a \qquad s\in S_a$$

$$\sum_{s\in S_a}(b'^s_a / b^s_a) y^s_a \le 1 \qquad a\in A$$

$$x^k_a \ge 0 \qquad a\in A,\quad k\in K$$

$$0 \le y^s_a \le b^s_a / b'^s_a \qquad a\in A,\quad s\in S_a$$

A lower bound and a capacity of ξ_a^s are changed

A coefficient of y^s_a is changed

An upper bound of y^s_a is changed

- $PLCB(b')$ = *LP*, solve it by mathematical programming software

Approximate method for *PLCB*

- Solve *PLCBL*(***b'***) iteratively
  - All variable $\tilde{y}_a^s$ do not always become 0 or 1.
  - It is possible that several $\tilde{y}_a^s$ of same arc is positive.
  - No guarantee that is satisfied the equation (6).

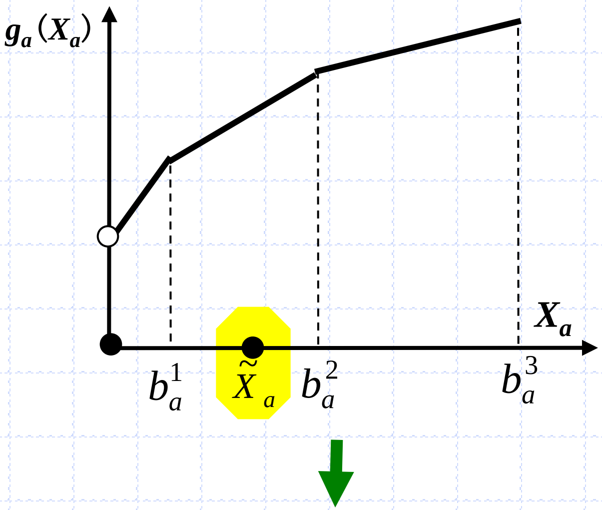
- Fix design variable to 0 or 1
- Design variable satisfies equation 6

$$\sum_{s \in S_a} y_a^s \leq 1 \qquad a \in A \qquad (6)$$

Approximate method for *PLCB*

- Fix the feasible design variables by linear relaxed solution $\tilde{X}_a$

$$\bar{y}_a^s = \begin{cases} 1 & b_a^{s-1} < \tilde{X}_a \le b_a^s \\ 0 & otherwise \end{cases} \qquad a \in A, s \in S_a$$



$y^1_a=0$, $y^2_a=1$, $y^3_a=0$

Capacity scaling procedure for *PLCB*

- Slove $PLCB(\bar{y})$ for obtaining upper bounds and feasible solutions of *PLCB*.

$PLCB(\bar{y})$

$$\min \sum_{a \in A} \sum_{s \in S_a} (c_a^s \xi_a^s + f_a^s \bar{y}_a^s)$$

Fixed y^s_a

s.t.

$$b_a^{s-1} \bar{y}_a^s \le \xi_a^s \le b_a^s \bar{y}_a^s \qquad a \in A, \quad s \in S_a$$

(2), (3), (4), (7)

- Multi-commodity flow problem with constant y^s_a

Capacity scaling procedure for *PLCB* Algorithm

1) set $\lambda \in (0,1]$, t_g, t_{max}. $b'^s_a := b^s_a$. $ub_{min:} = \infty$, $t := 0$.

2) $t := t+1$. Solve linear relaxed problem *PLCBL*(***b'***). Obtain the linear relaxed solutions $\tilde{X}_a$ and $\tilde{y}^s_a$

3) Change capacities $b'^s_a := \lambda b'^s_a y'^s_a + (1-\lambda) b'^s_a$

4) If t Mod $t_g = 0$ or $t = t_{max}$, go to 5), Otherwise go to 2) .

5) $$\bar{y}^s_a = \begin{cases} 1 & b^{s-1}_a < \tilde{X}_a \le b^s_a \\ 0 & otherwise \end{cases} \quad a \in A, s \in S_a$$

Solve *PLCB*($\bar{\mathbf{y}}$) and obtain $ub(t)$. If $ub_{min} > ub(t)$, $ub_{min} := ub(t)$

6) If $t = t_{max}$, stop , Otherwise, go to 2)

Numerical Experiments

- Data : Crainic's data for capacitated network design problem

a) Variable flow cost c_a^s : $c_a^s = c_a \alpha^{s-1}$,

c_a = Crainic's data , $\alpha = 0.7$

b) Fixed flow cost : f_a^1 = Crainic's data,

others → continue at the lower bound of segments

c) Number of Segment $|S_a|$: 3

d) Upper bound of s : $b_a^s = (2^s - 1)\, b_a / 2$,

b_a = Crainic's data

Numerical Experiments

- Error：(upper bound –lower bound)/lower bound
- Upper bound ：our method
- Lower bound: software CPLEX within 10 hours
- Software for *PLCBL*(***b'***)：GLPK ver.4.8
- Computing time ：max 10 hours
- Maximum iteration number ：100
- Search cycle for upper bound ：5
- Language ：C
- Computer ：Pentium 3.0GHz and 1Gbyte memories

Results

Names of Problems	Number of Nodes	Number of Arcs	Number of Commodities	The normal		The strong		The extended	
				Gaps(%)	Tims	Gaps(%)	Tims	Gaps(%)	Tims
1	25	100	10	52.3	3	7.3	16	4.7	308
2	25	100	10	22.2	3	5.6	30	2.8	185
3	25	100	10	2.7	3	0.8	12	0.0	287
4	25	100	30	36.6	13	6.9	262	5.3	4310
5	25	100	30	12.6	12	6.9	246	1.1	5880
6	25	100	30	3.8	11	3.6	104	0.2	3277
7	100	40	10	33.5	43	2.5	1064	1.0	7304
8	100	40	10	57.2	62	18.3	9154	14.6	33117
9	100	40	10	2.0	52	0.5	8223	0.0	27992
10	100	40	30	76.4	1068	24.0	15202	–	–
11	100	40	30	40.7	1498	19.7	18676	–	–
12	100	40	30	10.4	3029	9.9	17804	–	–

time: seconds

Conclusions

We consider the capacitated network design problem with piecewise linear costs.

- For the basic, strong and extend models, capacity scaling procedures are proposed.
- The effectiveness of formulations and solution procedures are showed by numerical experiments.

- Future tasks are
  - To develop a path flow formulation instead of an arc flow formulation, and develop path generation method
  - To develop the intermediate model between the basic and strong model, and the strong and extended model