A SUPPLY CHAIN NETWORK DESIGN PROBLEM WITH CONCAVE INVENTOY COSTS

S. Tei, N. Katayama, S. Yurimoto

Distribution and Logistics Systems, Ryutsu Keizai University 120 Hirahata, Ryugasaki, Ibaraki, Japan

Abstract
A supply chain network design problem consists of determining the location of plants and distribution centres, product assignments, transportation and distribution, and inventory levels at facilities, which minimize the total cost, including production, handling, inventory, transportation and distribution costs. Inventory costs are represented as concave functions and facility locations as binary variables, therefore a supply chain network design problem can be formulated as a large mixed integer optimization problem with concave functions. In this paper, the Lagrangian relaxation method is developed for the supply chain network design problem. The Lagrangian relaxation problem formed by relaxing flow conservation constraints is represented as continuous knapsack problems. Finally, for obtaining approximate solutions, a Lagrangian heuristic based on solving linear approximate network flow problems is presented.

Keywords:
supply chain, network design, multi-commodity flow, optimization

1 INTRODUCTION

Recently the global economy has suffered from the deep and long global recession which emerged from the U.S. In order to reduce global costs and adjust the volume of inventory and production, global companies must plan to urgently reconstruct and reduce the scale of their global supply chain networks.

The supply chain network is very large and complicated, and a plan to reconstruct and reduce the scale of a supply chain network is a very difficult problem. Constructing and solving a mathematical model of a supply chain network design are very useful tasks for reconstructing or reducing the network.

A supply chain network design problem consists of determining the location of plants and distribution centres, product assignments, transportation and distribution, and inventory levels in order to minimize the total cost, including production, handling, inventory, transportation and distribution costs. Inventory costs are represented as concave functions and facility locations as binary variables, therefore a supply chain network design problem can be formulated as a large mixed integer optimization problem with concave functions.

In this paper, the Lagrangian relaxation method is developed for the supply chain network design problem. The Lagrangian relaxation problem formed by relaxing multi-commodity flow conservation constraints can be decomposed into subproblems for each node and each product. The subproblems are represented as continuous knapsack problems and concave continuous knapsack problems, and they are easy solvable polynomially. Furthermore, for obtaining approximate solutions, the Lagrangian heuristic based on the solutions of the Lagrangian relaxation problems is developed by solving linear approximation problems.

In the earliest papers, Geoffrion-Graves [1] presents an algorithm based on Benders decomposition to solve a multi-commodity network design problem. Aikens [2], Geoffrion-Powers [3], Vidal-Goetschalckx [4] and others have expanded the model of Geoffrion and Graves.

Integrated supply chain network design problems including locations and inventory costs of distribution centres are developed. Daskin-Coullard-Shen [5] presents an inventory

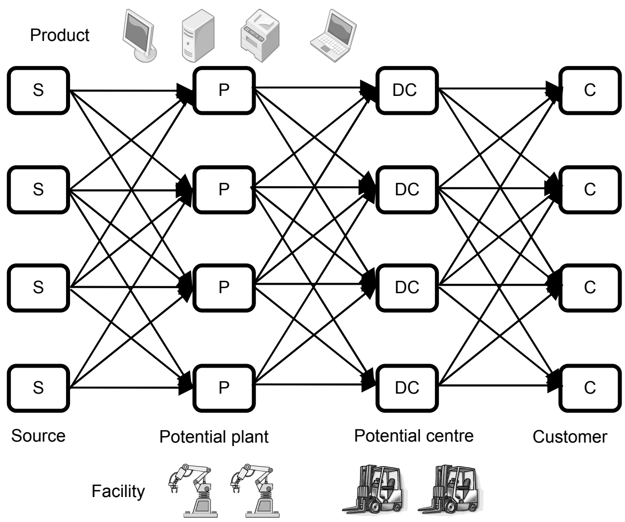

Figure1. Supply chain network design problem

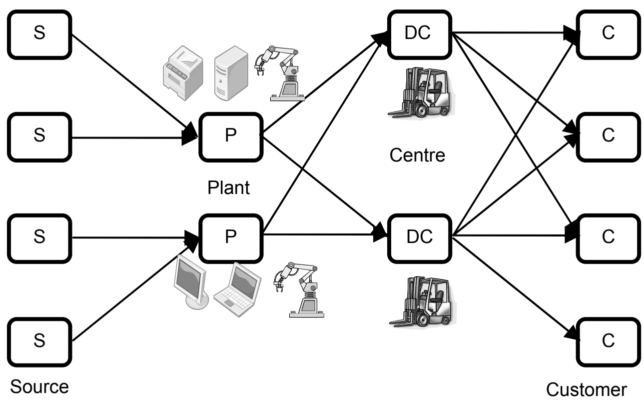

Figure 2. A solution of supply chain network

and location model, and Shu-Teo-Shen [6] develops a stochastic transportation and inventory model.

In a global supply chain network design problem, Vidal-Goetschalckx [7] presents a global model with transfer pricing and transportation cost allocation. Lowe-Wendell-Hu [8] develops a multi-screening approach for incorporating uncertainty about exchange rates and their risks.

The Lagrangian relaxation method [9] is one of general solution strategies for solving mathematical programming problems that permit us to decompose problems to exploit their particular structure. In the earliest papers, Held-Karp [10] applies a Lagrangian relaxation method and a subgradient procedure to travelling salesman problems. The Lagrangian relaxation method has been applied to a huge number of mathematical problems and yielded excellent results.

2 PROBLEM DESCRIPTION

2.1 Supply chain network design problem

Given potential product plants and potential distribution centres, customers, products and their demands, the supply chain network design problem is to determine product plants and distribution centres, product assignments, transportation and inventory on the network to minimize the total cost, including production, handling, inventory and transportation costs.

Figure 1 and Figure 2 show a supply chain network design problem and a solution of a supply chain network. The supply chain network configuration decision consists of determining which of the production plants and distribution centres to select, which of manufacturing and inventory facilities to be procured from for products at selected plants and centres, and product flows in the network.

Figure 3 shows a network representation of the supply chain network. Product plants, distribution centres and customers are represented as nodes, and transportation is represented as arcs. Product dummy source nodes are defined as origin nodes for product flows.

2.2 Inventory cost

Assuming that a distribution centre orders each product from the upstream product plant using (r, Q) policy with a certain service level, the frequency of orders and the order quantity at each centre are determined by the handling volume (which is the demand of downstream customers).

Let I_i^p denote the inventory holding cost of product *p* at centre *i,* and O_i^p denote the fixed order cost of product *p* from the upstream plant at centre *i.* N_i^- is the set of downstream customers of centre *i.* Let t_i^p denote the handling volume of product *p* at centre *I,* and n_i^p denote the number of orders from the upstream plant at centre *i*.

Then the average order size per order is t_i^p/n_i^p, and the average working inventory volume is $t_i^p/2n_i^p$. The total cost of orders and working inventory at centre *i* is given by

$$O_i^p n_i^p + I_i^p t_i^p / 2n_i^p \ . \tag{1}$$

The optimal value of n_i^p to minimize function (1) is obtained as

$$n_i^p = \sqrt{I_i^p t_i^p \big/ 2O_i^p} \ . \tag{2}$$

The corresponding total order and working inventory cost TC_i^p of associated node *i* and product *p* can be expressed as

$$TC_i^p = \sqrt{2 I_i^p O_i^p t_i^p} \ . \tag{3}$$

The volume of safety stock required to ensure that stockouts occur within a certain probability is given by

$$\alpha \sqrt{L \sum_{j \in N_i^-} \left(\sigma_{ij}^p\right)^2} \ , \tag{4}$$

where *α* is the safety stock factor, *L* is the lead time and σ_{ij}^p is the standard deviation of the demand for product *p* shipped from centre *i* to customer *j*. The corresponding inventory holding cost of the safety stock is given by

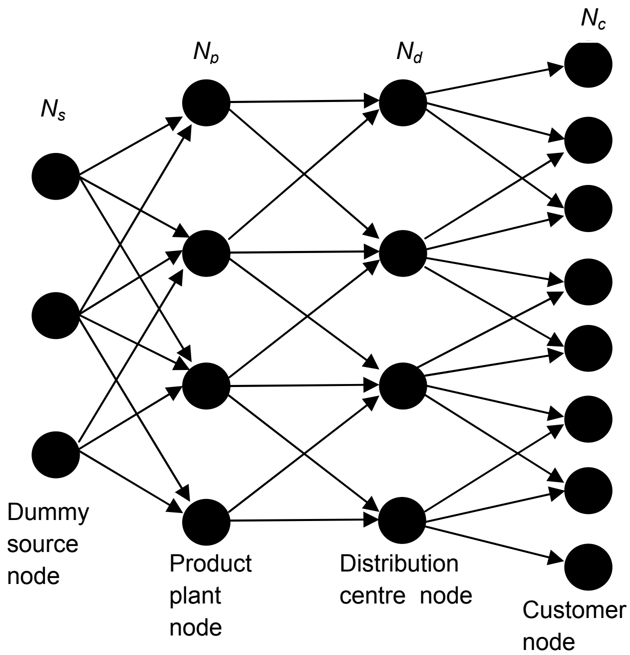

Figure 3. Network representation

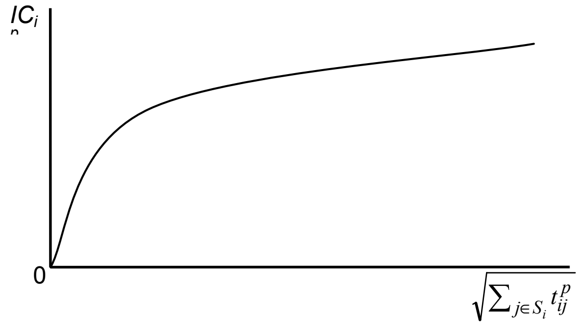

Figure 4. Inventory cost

$$I_i^p \alpha \sqrt{L \sum_{j \in N_i^-} \left(\sigma_{ij}^p\right)^2} \ . \tag{5}$$

Assuming that the ratio of the demand variance to the mean demand is the same value *β* for all downstream customers, i.e.

$$\left(\sigma_{ij}^p\right)^2 \Big/ t_{ij}^p = \beta \quad \forall j \in N_i^- \ , \tag{6}$$

where t_{ij}^p is the demand for product *p* shipped from centre *i* to customer *j*, the total inventory cost IC_i^p can be described as

$$IC_i^p = \left(\sqrt{2 I_i^p O_i^p} + I_i^p \alpha \sqrt{L\beta} \right) \sqrt{\sum_{j \in N_i^-} t_{ij}^p} \ . \tag{7}$$

As described above, the inventory cost can be represented as the square root functions of the handling volume at the centre. Figure 4 illustrates the total inventory cost IC_i^p at centre *i*.

2.3 Problem formulation

Let *G*=(*N*,*A*) be a supply chain network, *N* be the set of nodes and *A* be the set of arcs. *N* consists of the set of product dummy source nodes N_s, the set of production plants N_p, the set distribution centres N_d and the set of customers N_c. N_{pd} is $N_p \cup N_d$. N^+ is the set of nodes with the outdegree to other nodes, i.e. $N_s \cup N_p \cup N_d$, and N_i^+ is the set of nodes with the outdegree to node *i*. N_i^- is the set of nodes with the indegree from node *i*. *P* is the set of products produced and shipped in the supply chain network.

a_{ij}^p is the variable cost associated with node i and an arc from node i to node j of product p. If $i \in N_s$ and $j \in N_p$, then a_{ij}^p is the procurement cost of product p at plant j from the product dummy source node i. If $i \in N_p$ and $j \in N_d$, then a_{ij}^p is the production cost at plant i and the transportation cost of product p shipped from plant i to centre j. If $i \in N_d$ and $j \in N_c$, then a_{ij}^p is the handling cost at centre i and the transportation cost of product p shipped from centre i to customer j. b_i^p is a coefficient, $\sqrt{2I_i^p O_i^p} + I_i^p \alpha \sqrt{L\beta}$ associated with the inventory cost of product p at centre i, and the inventory cost of product p is proportional to the square root of the handling volume.

f_i is the fixed cost for selecting production plant or distribution centre i. g_i^p is the fixed cost for procuring a manufacturing or inventory facility for product p at node i. d_l^p is the demand of product p for customer l. C_i is the total capacity of production plant or distribution centre i. E_i^p is the capacity of a manufacturing or inventory facility at the selected plant and centre i for product p.

y_i is the binary decision variable for selecting a plant or centre i, which equals 1 if the production plant or distribution centre i is selected, and 0 otherwise. z_i^p is the binary decision variable for procuring a facility at plant or centre i, which equals 1 if the manufacturing or inventory facility of product p at node i is procured, and 0 otherwise. x_{ij}^{pl} is a continuous decision variable denoting the transportation volume of product p shipped from node i to node j for customer l. if $i \in N_d$, then $t_{ij}^p = \sum_{j \in N_c} x_{ij}^{pj}$. s^p is a product dummy source node of product p.

The supply chain network design problem can be formulated as a mixed integer and nonlinear programming problem P as follows:

(P)

$$\text{minimize} \sum_{i\in N^+}\sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c} a_{ij}^p x_{ij}^{pl} + \sum_{i\in N_d}\sum_{p\in P} b_i^p \sqrt{\sum_{j\in N_i^-} x_{ij}^{pj}} + \sum_{i\in N_{pd}}\left\{ f_i y_i + \sum_{p\in P} g_i^p z_i^p \right\} \tag{8}$$

$$\text{subject to} \sum_{i\in N_n^+} x_{in}^{pl} - \sum_{j\in N_n^-} x_{nj}^{pl} = \begin{cases} -d_l^p & \text{if } n = s^p \\ d_l^p & \text{if } n = l \\ 0 & \text{otherwise} \end{cases} \quad \forall n\in N, l\in N_c, p\in P \tag{9}$$

$$\sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl} \le C_i y_i \quad \forall i\in N_{pd} \tag{10}$$

$$\sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl} \le E_i^p z_i^p \quad \forall i\in N_{pd}, p\in P \tag{11}$$

$$\sum_{n\in N_i^+}\sum_{l\in N_c} x_{ni}^{pl} \le E_i^p z_i^p \quad \forall i\in N_p, p\in P \tag{12}$$

$$z_i^p \le y_i \quad \forall i\in N^+, p\in P \tag{13}$$

$$0 \le x_{ij}^{pl} \le d_l^p z_i^p \quad \forall l\in N_c, j\in N_i^-, i\in N_{pd}, p\in P \tag{14}$$

$$0 \le x_{ni}^{pl} \le d_l^p z_i^p \quad \forall l\in N_c, n\in N_i^+, i\in N_p, p\in P \tag{15}$$

$$y_i \in \{0,1\} \quad \forall i\in N_{pd} \tag{16}$$

$$z_i^p \in \{0,1\} \quad \forall i\in N_{pd}, p\in P. \tag{17}$$

The objective function (8) is the total cost in the network, and should be minimized. The first term is production costs and handling costs at production plants and distribution centres, and transportation costs of shipping to distribution centres and customers, or procurement costs for products. The second term is the nonlinear function representing inventory costs at distribution centres. The third term is the fixed costs of selecting production plants and distribution centres, and the last term is the fixed costs of procuring production and inventory facilities. Constraints (9) represent the multi-commodity flow conservation constraints which secure a demand flow from a product dummy source node to a customer via a plant and a centre for each product. Constraints (10) represent the capacity constraints at a production plant or a distribution centre. If plant or centre i is selected, y_i=1, the products can be produced or handled less than or equal to capacity C_i totally, otherwise none. Constraints (11) represent the facility capacity constraints at a manufacturing or inventory facility. If the facility of product p at plant or centre i is procured, z_i^p=1, the product volume which can be produced or handled is less than or equal to facility capacity E_i^p, otherwise none. Constraints (12) represent the facility capacity constraints for a manufacturing facility. If the facility of product p at plant i is procured, z_i^p=1, and the products from product dummy source node n which can be procured are less than or equal to facility capacity E_i^p, otherwise none. Constraints (13) state that if plant or centre i is not selected, then no facility at node i should be procured. Constraints (14) and (15) state that if the facility of product p at node i is procured, then the maximum flow of product p for customer l via node i and node j is its demand and the minimum is 0, otherwise 0. Constraints (16) and (17) are binary requirements.

3 A LAGRANGIAN RELAXATION METHOD

3.1 A Lagrangian Relaxation Problem

Using the Lagrangian multiplier vector $\boldsymbol{v}=(v_n^{pl})$ associated with constraints (9) and adding it to the objective function (8), the following Lagrangian relaxation problem LG can be formed.

(LG)

$$\text{minimize} \sum_{p\in P}\left\{\sum_{l\in N_c} d_l^p\left(v_l^{pl} - v_{s^p}^{pl}\right)\right\} + \sum_{i\in N^+}\sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c}\left(a_{ij}^p + v_i^{pl} - v_j^{pl}\right)x_{ij}^{pl} + \sum_{i\in N_d}\sum_{p\in P} b_i^p\sqrt{\sum_{j\in N_i^-} x_{ij}^{pj}} + \sum_{i\in N_{pd}}\left\{f_i y_i + \sum_{p\in P} g_i^p z_i^p\right\} \tag{18}$$

subject to (10)-(17)

As LG is the relaxation problem of P, the optimal value of LG is a lower bound of the optimal value of P, and the optimal solution of LG may not a feasible solution of P.

Given the Lagrangian multiplier vector $\boldsymbol{v}$, the first term of the objective function (18) is a constant term. Therefore, LG can be decomposed into the following subproblem LG_i^p for each node $i \in N_p$, and LG_i^d for each node $i \in N_d$.

(LG_i^p)

$$\text{minimize} \sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c}\left(a_{ij}^p + v_i^{pl} - v_j^{pl}\right)x_{ij}^{pl} + \sum_{p\in P}\sum_{n\in N_i^+}\sum_{l\in N_c}\left(a_{ni}^p + v_n^{pl} - v_i^{pl}\right)x_{ni}^{pl} + f_i y_i + \sum_{p\in P} g_i^p z_i^p \tag{19}$$

$$\sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl} \le C_i y_i \tag{20}$$

$$\sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl} \le E_i^p z_i^p \qquad \forall p\in P \tag{21}$$

$$\sum_{n\in N_i^+}\sum_{l\in N_c} x_{ni}^{pl} \le E_i^p z_i^p \qquad \forall p\in P \tag{22}$$

$$z_i^p \le y_i \qquad \forall\, p\in P \tag{23}$$

$$0 \le x_{ij}^{pl} \le d_l^p z_i^p \qquad \forall l\in N_c, j\in N_i^-, p\in P \tag{24}$$

$$0 \le x_{ni}^{pl} \le d_l^p z_i^p \qquad \forall l\in N_c, n\in N_i^+, p\in P \tag{25}$$

$$y_i \in \{0,1\} \tag{26}$$

$$z_i^p \in \{0,1\} \qquad \forall\, p\in P \tag{27}$$

(LG_i^d)

$$\textit{minimize} \quad \sum_{p\in P}\sum_{j\in N_i^-}\left(a_{ij}^p + v_i^{pj} - v_j^{pj}\right)x_{ij}^{pj} + \sum_{p\in P} b_i^p \sqrt{\sum_{j\in N_i^-} x_{ij}^{pj}} + f_i y_i + \sum_{p\in P} g_i^p z_i^p \tag{28}$$

$$\textit{subject to} \quad \sum_{p\in P}\sum_{j\in N_i^-} x_{ij}^{pj} \le C_i y_i \tag{29}$$

$$\sum_{j\in N_i^-} x_{ij}^{pj} \le E_i^p z_i^p \qquad \forall p\in P \tag{30}$$

$$0 \le x_{ij}^{pj} \le d_i^p z_i^p \qquad \forall j\in N_i^-, p\in P \tag{31}$$

and (23), (26), (27)

$\sum_{j\in N_i^-} x_{ij}^{pj}$ can be substituted for $\sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl}$ for $i\in N_d$.

LG_i^p and LG_i^d can be decomposed into two subproblems in the case of y_i=0 and y_i=1. Obviously, when y_i=0, the optimal solution is $\boldsymbol{z}$=**0** and $\boldsymbol{x}$=**0**, and the optimal value of the objective functions (19) and (28) is 0. When y_i=1, LG_i^p and LG_i^d can be rewritten as the following problems LG_i^{p1} and LG_i^{d1}.

(LG_i^{p1})

$$\textit{minimize} \quad \sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c}\left(a_{ij}^p + v_i^{pl} - v_j^{pl}\right)x_{ij}^{pl} + \sum_{p\in P}\sum_{n\in N_i^+}\sum_{l\in N_c}\left(a_{ni}^p + v_n^{pl} - v_i^{pl}\right)x_{ni}^{pl} + \sum_{p\in P} g_i^p z_i^p + f_i \tag{32}$$

$$\textit{subject to} \sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl} \le C_i \tag{33}$$

and (21), (22), (24), (25), (27)

(LG_i^{d1})

$$\textit{minimize} \quad \sum_{p\in P}\sum_{j\in N_i^-}\left(a_{ij}^p + v_i^{pj} - v_j^{pj}\right)x_{ij}^{pj} + \sum_{p\in P} b_i^p \sqrt{\sum_{j\in N_i^-} x_{ij}^{pj}} + \sum_{p\in P} g_i^p z_i^p + f_i \tag{34}$$

$$\textit{subject to} \sum_{p\in P}\sum_{j\in N_i^-} x_{ij}^{pj} \le C_i \tag{35}$$

and (27), (30), (31)

Furthermore, using the nonnegative Lagrangian multiplier u_i associated with capacity constraints (33) and (35), and adding them to the objective functions (32) and (34), the following Lagrangian relaxation problems LG_i^{p1c} and LG_i^{d1c} can be formed.

(LG_i^{p1c})

$$\textit{minimize} \quad \sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c}\left(a_{ij}^p + v_i^{pl} - v_j^{pl} + u_i\right)x_{ij}^{pl} + \sum_{p\in P}\sum_{n\in N_i^+}\sum_{l\in N_c}\left(a_{ni}^p + v_n^{pl} - v_i^{pl}\right)x_{ni}^{pl} + \sum_{p\in P} g_i^p z_i^p - C_i u_i + f_i \tag{36}$$

subject to (21), (22), (24), (25), (27)

(LG_i^{d1c})

$$\textit{minimize} \quad \sum_{p\in P}\sum_{j\in N_i^-}\left(a_{ij}^p + v_i^{pj} - v_j^{pj} + u_i\right)x_{ij}^{pj} + \sum_{p\in P} b_i^p \sqrt{\sum_{j\in N_i^-} x_{ij}^{pj}} + \sum_{p\in P} g_i^p z_i^p - C_i u_i + f_i \tag{37}$$

subject to (27), (30), (31)

As LG_i^{p1c} and LG_i^{d1c} have only one Lagrangian multiplier u_i, the optimal value of u_i can be easily obtained by a binary search.

Given the Lagrangian multiplier u_i, the last two terms of the objective functions (36) and (37) are constant terms. Then LG_i^{p1c} and LG_i^{d1c} can be decomposed into the following subproblem LG_{ip}^{p1c} and LG_{ip}^{d1c} for each product $p\in P$.

(LG_{ip}^{p1c})

$$\textit{minimize} \quad \sum_{j\in N_i^-}\sum_{l\in N_c}\left(a_{ij}^p + v_i^{pl} - v_j^{pl} + u_i\right)x_{ij}^{pl} + \sum_{n\in N_i^+}\sum_{l\in N_c}\left(a_{ni}^p + v_n^{pl} - v_i^{pl}\right)x_{ni}^{pl} + g_i^p z_i^p \tag{38}$$

$$\textit{subject to} \sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl} \le E_i^p z_i^p \tag{39}$$

$$\sum_{n\in N_i^+}\sum_{l\in N_c} x_{ni}^{pl} \le E_i^p z_i^p \tag{40}$$

$$0 \le x_{ij}^{pl} \le d_l^p z_i^p \qquad \forall l\in N_c, j\in N_i^- \tag{41}$$

$$0 \le x_{ni}^{pl} \le d_l^p z_i^p \qquad \forall l\in N_c, n\in N_i^+ \tag{42}$$

$$z_i^p \in \{0,1\} \tag{43}$$

(LG_{ip}^{d1c})

$$\textit{minimize} \quad \sum_{j\in N_i^-}\left(a_{ij}^p + v_i^{pj} - v_j^{pj} + u_i\right)x_{ij}^{pj} + b_i^p \sqrt{\sum_{j\in N_i^-} x_{ij}^{pj}} + g_i^p z_i^p \tag{44}$$

$$\textit{subject to} \quad \sum_{j\in N_i^-} x_{ij}^{pj} \le E_i^p z_i^p \tag{45}$$

$$0 \le x_{ij}^{pj} \le d_j^p z_i^p \qquad \forall j\in N_i^- \tag{46}$$

and (43)

LG_{ip}^{p1c} and LG_{ip}^{d1c} have one binary variable z_i^p and can be decomposed into two subproblems in the case of z_i^p=0 and z_i^p=1. Obviously, when z_i^p=0, the optimal solution is $\boldsymbol{x}$=**0**, and the optimal value of the objective functions (38) and (44) is 0. When z_i^p=1, these problems can be rewritten as the following problems LG_{ip}^{p1c1} and LG_{ip}^{d1c1}.

(LG_{ip}^{p1c1})

$$\phi_{ip}^{p} = minimize \quad \sum_{j\in N_i^-}\sum_{l\in N_c}\left(a_{ij}^{p} + v_i^{pl} - v_j^{pl} + u_i\right)x_{ij}^{pl} + \sum_{n\in N_i^+}\sum_{l\in N_c}\left(a_{ni}^{p} + v_n^{pl} - v_i^{pl}\right)x_{ni}^{pl} + g_i^{p} \tag{47}$$

$$subject\ to \sum_{l\in N_c}\sum_{j\in N_i^-} x_{ij}^{pl} \le E_i^{p} \tag{48}$$

$$\sum_{n\in N_i^+}\sum_{l\in N_c} x_{ni}^{pl} \le E_i^{p} \tag{49}$$

$$0 \le x_{ij}^{pl} \le d_l^{p} \qquad \forall l\in N_c, j\in N_i^- \tag{50}$$

$$0 \le x_{ni}^{pl} \le d_l^{p} \qquad \forall l\in N_c, n\in N_i^+ \tag{51}$$

(LG_{ip}^{d1c1})

$$\phi_{ip}^{d} = minimize \quad \sum_{j\in N_i^-}\left(a_{ij}^{p} + v_i^{pj} - v_j^{pj} + u_i\right)x_{ij}^{pj} + b_i^{p}\sqrt{\sum_{j\in N_i^-} x_{ij}^{pj}} + g_i^{p} \tag{52}$$

$$subject\ to \sum_{j\in N_i^-} x_{ij}^{pj} \le E_i^{p} \tag{53}$$

$$0 \le x_{ij}^{pj} \le d_j^{p} \qquad \forall j\in N_i^- \tag{54}$$

where Φ_{ip}^{p} and Φ_{ip}^{d} are the optimal values of the objective functions.

LG_{ip}^{p1c1} is divided into two continuous knapsack problems with respect to the first term of objective function (47) with constraints (48) and (50), and the second term of objective function (47) with constraints (49) and (51). These continuous knapsack problems can be solved easily by sorting of coefficients of ***x*** in (47).

LG_{ip}^{d1c1} is a concave continuous knapsack problem, and can be solved in polynomial time [11]. The objective function (52) is concave, therefore the optimal solution is one of the extreme points on the function. Figure 5 illustrates the extreme points on the function.

Let $h_j = a_{ij}^{p} + v_i^{pj} - v_j^{pj} + u_i$, and permutation ***m*** satisfies the following equation.

$$h_{m_1} \le h_{m_2} \le \cdots \le h_{mN_i^-} \tag{55}$$

$$H^{q} = \sum_{k=1}^{q} h_{m_k} d_{m_k}^{p} + b_i^{p}\sqrt{\sum_{k=1}^{q} d_{m_k}^{p}} \qquad \forall 1 \le q < \bar{q} \tag{56}$$

$$H^{\bar{q}} = \sum_{k=1}^{\bar{q}-1} h_{m_k} d_{m_k}^{p} + \left(E_i^{p} - \sum_{k=1}^{\bar{q}-1} d_{m_k}^{p}\right)h_{m_k} + b_i^{p}\sqrt{E_i^{p}} \tag{57}$$

where $\bar{q}$ is the minimum index q which satisfies $\sum_{k=1}^{q} d_{m(k)}^{p} \ge E_i^{p}$.

Let $H = min_{1\le q\le \bar{q}} H_{ip}^{q}$. If $H \le 0$, then the optimal solution is ***x*=0**, or otherwise as follows:

$$x_{im_k}^{pm_k} = \begin{cases} d_{m_k}^{p} & k = 1,\cdots,\bar{q}-1 \\ E_i^{p} - \sum_{k=1}^{\bar{q}-1} d_{m_k}^{p} & k = \bar{q} \\ 0 & k > \bar{q}. \end{cases} \tag{58}$$

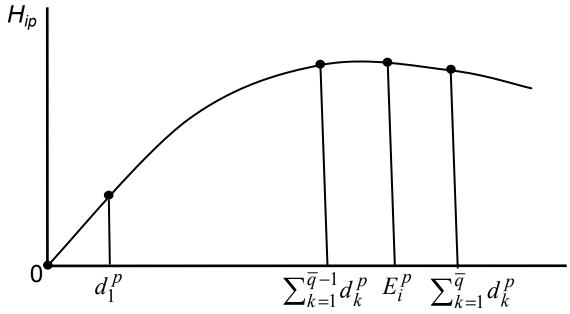

Figure 5. Extreme points

As mentioned above, the Lagrangian relaxation problem *LG* can be solved polynomially, and the lower bound *LB* of *P* can be obtained as follows:

$$LB = \sum_{i\in N_p}\min\left\{0, \sum_{p\in P}\min\left(0,\phi_{ip}^{p}\right) - C_i u_i + f_i\right\} + \sum_{i\in N_d}\min\left\{0, \sum_{p\in P}\min\left(0,\phi_{ip}^{d}\right) - C_i u_i + f_i\right\} + \sum_{p\in P}\left\{\sum_{l\in N_c} d_l^{p}\left(v_l^{pl} - v_{s^p}^{pl}\right)\right\}. \tag{59}$$

3.2 A SUBGRADIENT METHOD

For choosing the appropriate value of ***u***, a binary search is applied for each LG_i^{p1c} and LG_i^{d1c}. The range of the optimal value of u_i is from to $-min(a_{ij}^{p}+v_i^{pl}-v_j^{pl}+u_i)$. Given each u_i at the binary search, LG_i^{p1c} and LG_i^{d1c} are solved. And the optimal value of u_i such as minimizing the optimal value of LG_i^{p1c} and LG_i^{d1c} is found.

For choosing the appropriate value of ***v*** approximately, the standard subgradient optimization procedure is applied. This is an iterative procedure, which uses the current multipliers ***v***, the current lower bound and an upper bound, in order to compute the new multipliers ***v*** used in the next iteration. The subgradient ***w*** of ***v*** at the current solution $\bar{\boldsymbol{x}}$ of *LG* can be defined as follows:

$$w_n^{pl} = \begin{cases} -d_l^{p} - \sum_{i\in N_n^+}\bar{x}_{in}^{pl} + \sum_{j\in N_n^-}\bar{x}_{nj}^{pl} & if\ n = s^{p} \\ d_l^{p} - \sum_{i\in N_n^+}\bar{x}_{in}^{pl} + \sum_{j\in N_n^-}\bar{x}_{nj}^{pl} & if\ n = l \\ -\sum_{i\in N_n^+}\bar{x}_{in}^{pl} + \sum_{j\in N_n^-}\bar{x}_{nj}^{pl} & otherwise \end{cases} \tag{60}$$

$$\forall n\in N, l\in N_c, p\in P.$$

By using a step size s^t in iteration t, the new set of multipliers are given by

$$v_n^{pl} := v_n^{pl} + s^{t} w_n^{pl} \qquad \forall n\in N, l\in N_c, p\in P. \tag{61}$$

The step size s^t is selected according to

$$s^{t} := \rho(upper\ bound - LB))/|\boldsymbol{w}|^{2}, \tag{62}$$

where ρ is a scalar, which is initially equal to 1 and is reduced every certain iteration number.

4 LAGRANGIAN HEURISTIC

A Lagrangian heuristic is an approximate solution method, and derives feasible solutions from Lagrangian relaxation solutions. The optimal solutions $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{z}}$, $\bar{\boldsymbol{y}}$ of *LG* and lower bound *LB* of *P* are obtained by solving *LG*. But $\bar{\boldsymbol{x}}$ may not be feasible for *P,* due to relaxation of flow conservation constraints. The Lagrangian heuristic is presented based on $\bar{\boldsymbol{x}}$, by using a linear approximation near $\bar{\boldsymbol{x}}$.

The objective function of *P* involves nonlinear functions. *P* is not a linear programming problem, therefore it is hard to solve *P* by a mathematical programming solver such as CPLEX. Therefore the linear approximation of the square root function near $\bar{\boldsymbol{x}}$ is used as follows:

$$\sqrt{\sum_{j\in N_i^-} x_{ij}^{pj}} \approx \frac{1}{2A(\bar{x})}\sum_{j\in N_i^-} x_{ij}^{pj} - \frac{1}{2}A(\bar{x}) \tag{63}$$

where $A(\bar{x}) = \sqrt{\sum_{j\in N_i^-} \bar{x}_{ij}^{pj}}$.

The linear approximation problem near $\bar{\boldsymbol{x}}$ at $\bar{\boldsymbol{z}}$ and $\bar{\boldsymbol{y}}$ of *P* is formed as the following linear programming problem $LA(\bar{\boldsymbol{x}})$.

($LA(\bar{\boldsymbol{x}})$)

minimize

$$\sum_{i\in N^+}\sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c} a_{ij}^p x_{ij}^{pl} + \sum_{i\in N_d}\sum_{p\in P}\sum_{j\in N_i^-} \frac{b_i^p}{2A(\bar{x})} x_{ij}^{pj} \tag{64}$$

$$-\sum_{i\in N_d}\sum_{p\in P}\frac{b_i^p A(\bar{x})}{2} + \sum_{i\in N_{pd}}\left\{ f_i \bar{y}_i + \sum_{p\in P} g_i^p \bar{z}_i^p \right\}$$

$$\sum_{p\in P}\sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl} \le C_i \bar{y}_i \qquad \forall i\in N_{pd} \tag{65}$$

$$\sum_{j\in N_i^-}\sum_{l\in N_c} x_{ij}^{pl} \le E_i^p \bar{z}_i^p \qquad \forall i\in N_{pd}, p\in P \tag{66}$$

$$\sum_{n\in N_i^+}\sum_{l\in N_c} x_{ni}^{pl} \le E_i^p \bar{z}_i^p \qquad \forall i\in N_p, p\in P \tag{67}$$

$$0 \le x_{ij}^{pl} \le d_l^p \bar{z}_i^p \qquad \forall l\in N_c, i\in N_{pd}, j\in N_i^-, p\in P \tag{68}$$

$$0 \le x_{ni}^{pl} \le d_l^p \bar{z}_i^p \qquad \forall l\in N_c, i\in N_p, n\in N_i^+, p\in P \tag{69}$$

$LA(\bar{\boldsymbol{x}})$ can be solved by a mathematical programming solver. $\bar{\boldsymbol{x}}$ is an initial solution for $LA(\bar{\boldsymbol{x}})$. Let the solution of $LA(\bar{\boldsymbol{x}})$ denote $\bar{\boldsymbol{x}}^1$, and $\bar{\boldsymbol{x}}^k$ denote the solution at the *k*th iteration. $LA(\bar{\boldsymbol{x}}^k)$ is solved iteratively until $\bar{\boldsymbol{x}}^k$ converges. If $LA(\bar{\boldsymbol{x}})$ is not feasible, a feasible solution of *P* cannot be obtained from $\bar{\boldsymbol{x}}$. In the subgradient method, various Lagrangian relaxation solutions are obtained at iterations, therefore various initial feasible solutions and various convergent feasible solutions can be obtained.

The whole algorithm of the Lagrangian relaxation method for the supply chain network design problem is described as follows:

[Step1] Given $\boldsymbol{v}$, t:=0, ite_{max}:=the iteration of maximum numbers, LB:=0, UB:=∞.

[Step2] Solve *LG* with $\boldsymbol{v}$ and obtain $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{z}}$, $\bar{\boldsymbol{y}}$.

[2-1] Set $\boldsymbol{u}$ at the binary search, and solve LG_i^{p1c} and LG_i^{d1c}.

[2-2] Find the optimal value of $\boldsymbol{u}$ and obtain LB^t.

[2-3] If LB^t>LB, then LB:=LB^t

[Step3] Solve $LA(\bar{\boldsymbol{x}})$ at $\bar{\boldsymbol{z}}$ and $\bar{\boldsymbol{y}}$.

[3-1] If $LA(\bar{\boldsymbol{x}})$ is feasible, then obtain the convergent feasible solution and upper bound UB^t.

[3-2] If UB^t<UB, then UB:=UB^t.

[Step4] Update $\boldsymbol{v}$ by the subgradient method.

[Step5] t:=t+1. If t>ite_{max}, then stop, else go to Step2.

5 CONCLUSION

In this paper, the Lagrangian relaxation method is developed for the supply chain network design problem. The supply chain network design problem is formulated as a mixed integer and nonlinear programming problem, and the Lagrangian relaxation problem formed by relaxing multi-commodity flow conservation constraints can be decomposed into subproblems for each node and each product. The subproblems are represented as continuous knapsack problems, and are solvable polynomially. Finally, for obtaining approximate solutions, the Lagrangian heuristic based on the solutions of the Lagrangian relaxation problems is developed by solving linear approximation problems.

Numerical experiments to evaluate the effectiveness of the Lagrangian relaxation method have not been presented yet. This research is underway to evaluate by numerical experiments and apply to real world problems.

References

[1] Geoffrion, A.M, Graves, G.W., Multicommodity distribution system design by Benders decomposition, Management Science, 1974, 20, 822-44.

[2] Aikens, C.H., Facility planning models for distribution planning, European Journal of Operational Research, 1985, 22, 263–279.

[3] Geoffrion, A.M. Powers R.F., Twenty years of strategic distribution system design: An evolutionary perspective, Interfaces, 1995, 25, 105-127.

[4] Vidal, C.J., Goetschalckx, M., Strategic production-distribution models: A critical review with emphasis on global supply chain models, European Journal of Operational Research, 1997, 98, 1-18.

[5] Daskin, M.S., Coullard C., Shen Z.J., An Inventory-location model: Formulation, solution algorithm and computational results, Annals of Operations Research, 2002, 110, 83-106.

[6] Shu, J., Teo, C.P., Shen, M., Stochastic transportation-inventory network design problem, Operations Research, 2005, 53, 48-60.

[7] Vidal, C.J., Goetschalckx, M., A global supply chain model with transfer pricing and transportation cost allocation, European Journal of Operational Research, 2001, 129, 134-158.

[8] Lowe, T.J., Wendell R. E., Hu G., Screening location strategies to reduce exchange rate risk, European Journal of Operations Research, 2002, 136, 573-590.

[9] Fisher, M.L., The Lagrangian relaxation method for solving integer programming problems, Management Science, 1981, 27, 1-18.

[10] Held, M. Karp,R.M, The travelling salesman problem and minimum spanning trees, Operation Research 1970, 18, 1138-1162.

[11] Kubo, M. Kasugai, H., A Lagrangian approach to the facility location problem with concave costs, Journal of Operations Research Society of Japan, 1991, 34, 335-353.